Appendix 2
(Under submission)

NSG-1625

10p.

A MARKOV DECISION MODEL FOR OVERHAUL OF REPAIRABLE REDUNDANT SYSTEMS

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In order to meet high reliability requirements, complex machines are often designed with a k-out-of-n component configuration, in which the system is operable so long as at least k of its n identical components work. Such reliable machines are often composed of repairable components and are maintained through scheduled, periodic overhauls, during which all failed parts are replaced by good ones. We develop a Markov decision model to determine the optimal inventories of repairable spare parts for such a system and simultaneously determine the optimal repair policy when there is a choice at each overhaul among a set of repair rates. The repair policy may be state-dependent; a policy might use a fast repair rate when q or fewer spares are on hand at the end of an overhaul, and use a slow repair rate otherwise. The objective is to minimize total long-run expected costs of spares, stockouts and repairs. A simple example is included to illustrate the model and its solution through linear programming.

[1]This research was supported by grant NSG 1625 from the NASA-Langley Research Center and by the College of William and Mary.

(NASA-CR-179777) A MARKOV DECISION MODEL FOR OVERHAUL OF REPAIRABLE REDUNDANT SYSTEMS (College of William and Mary) 10 p

N87-70028

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1. INTRODUCTION

Complex machines which must perform very reliably are often designed so that they can function even when some components have failed, employing a k-out-of-n component configuration. Such machines are often scheduled for periodic overhauls at a maintenance center, during which all failed parts are replaced by good ones to renew the machine's reliability. In general, periodic overhauls are advisable when emergency repair during the time interval between overhauls is impossible or very costly, as will often be the case for systems on submarines, aircraft, helicopters, and space vehicles.

A typical application of the model involves ultra-reliable avionics control systems for commercial aircraft, presently being developed by NASA. The avionics system will consist of reliable, repairable, expensive modules, arranged in a k-out-of-n configuration. For example, the control system might be operable if 3 out of 5 CPU's were operating. Demands for parts at periodic overhauls will be met by inventories stocked at a maintenance center.

The machines must leave the maintenance center in "good-as-new" condition in order to meet reliability requirements. Therefore, if a needed part or parts are not available in current inventory, the overhaul cannot be completed without incurring extra cost. Required parts which are out of stock must be obtained by some emergency procedure, such as immediate repair of the part. Stockout costs will vary, depending upon the number of parts required.

A Markov decision model to determine the optimal repairable parts inventory and repair strategy for such a maintenance center is developed in this paper. Total long-run expected shortage costs, repair costs, and holding costs are minimized for a machine containing a single system of redundant parts. Transition probabilities are calculated for each possible state and repair rate, and the optimal spare parts inventory and repair strategy is determined through linear programming. A simple example is included to illustrate the model.

2. RELATED MODELS IN THE LITERATURE

There is very little literature on the problem of optimal maintenance center inventories. In a recent survey by Nahmias [2] of the repairable inventory literature, there were no maintenance center models cited. In all of the models he reviewed, a demand for a part was recognized as soon as the part failed. In a maintenance center model, demands for parts occur at periodic overhauls, where several parts may be demanded simultaneously and all demands must be met in order to complete the overhaul. Schaefer [4] found optimal maintenance center inventories for complex machines under a job-completion criterion, where a single stockout penalty was assessed if the overhaul could not be completed because of shortage. The model handled machines with several types of parts, but exact results were obtained only when part failures were low and there was only one part of each type. Lawrence and Schaefer [3] found optimal inventories for a set of independent k-out-of-n systems under a constraint on total investment, but the repair rate for each type of part was constant, so that the decision variables were the initial number of spares of each type. The model presented here, a rudimentary version of which appeared in Schaefer [5], assesses a different stockout penalty for each number of parts missing and sums to determine the expected stockout costs. Using a Markov decison process model, the optimal spares inventory and repair shop strategy are determined.

3. THE MODEL AND ITS LINEAR PROGRAMMING FORMULATION

Consider a maintenance center serving a set of identical machines containing a single k-out-of-n system of independent identical components. We assume that each machine has an identical workload and that one machine arrives for overhaul each day. We seek the initial inventory of spare parts and repair strategy which minimizes expected stockout costs, repair costs and holding costs. We assume that each machine arrives for overhaul in functioning condition so that at least k parts are working. That is, the machine is so reliable that the probability that more than n-k parts have failed is negligible, surely a reasonable assumption since redundancy is employed to ensure high reliability. Each failed part is immediately sent to a repair shop, from which it eventually returns to replenish the stock of spares at the maintenance center. The total number of spares on hand and undergoing repair is a constant, s, which is a decision variable in the model. A stockout situation will occur if demand for spares at an overhaul exceeds the number of spares presently on hand. We assume that even if the part is stocked out, the machine is released from overhaul without appreciable delay, although a shortage penalty is assessed. Thus excess demands are not backordered.

We assume that after each overhaul the maintenance center can choose the repair rate to be employed by the repair shop until the next overhaul, presumably by using overtime or additional repairmen. The number of initial spares and the repair rate are traded off against each other; a small number of spares repaired rapidly may be as effective as a large number of spares repaired at a slow rate. In fact, if repairs were instantaneous, no spares at all would be required.

We will assume that using a faster repair rate costs more than a slow one. Otherwise, it would never be optimal to use a slow repair rate. Although faster repair rates are more costly, if they are utilized, the expected repair time per spare is decreased so the chances of stockout penalty at the next overhaul are diminished. The optimal repair strategy might be to choose a slow repair rate at all times when there are several spares on hand and to use a faster rate whenever current spares inventory is low, so as to decrease the probabliity of stockout at the next overhaul. Of course, the optimal number of spares and repair strategy will depend upon the relative costs of stockouts, repairs and inventory.

We model the inventory process as a Markov chain, with time intervals of one day and states representing the number of spares available at the maintenance center at the end of the day. Stockouts are denoted by negative state values. Thus the possible states are $m-n,\ldots,s$, where s is the number of spares initially stocked.

We introduce the following notation:

H = daily holding cost of a spare;

μ_r = the rth exponential repair rate, $1 \leq r \leq R$;

K_r = expected cost per day of repairing spares at rate μ_r;

λ = constant failure rate of each component in failures/hour;

T = cycle time between overhauls, in hours;

$a = 1 - \exp(-\lambda T)$ = probability that a single part has failed during the cycle;

$b_r = 1 - \exp(-\mu_r)$ = probability that a part which was at the repair shop at the end of one day returns to the center by the end of the next day using repair rate μ_r;

L_i = penalty cost for having i spares on hand, i<0 (emergency repair cost);

$Z_r(t,u)$ = conditional probability that t spares return to the maintenance center from the repair shop on a day that begins with u in the repair shop, using repair rate μ_r, $0 \leq t \leq u \leq s$;

$P_s(r)$ = matrix of transition probabilities with elements P_{ij}, representing the probability that j spares are on hand at the end of a day, given that i spares were on hand at the end of the previous day, the repair rate was μ_r and there were s spares on hand initially.

The probability v_m that m parts are demanded at an overhaul, $0 \leq m \leq n-k$, is $v_m = \binom{n}{m} a^m (1-a)^{n-m}$. Also, $Z_r(t,u) = \binom{u}{t} b_r^{\,t} (1-b_r)^{u-t}$. Then $P_{ij}(r)$ for a given value of s may be calculated as

$$P_{ij}(r) = \sum_{m=\max\{0,i-j\}}^{\min \Sigma\{s-j,n-k\}} v_m Z_r(m+j-i,s-i) \text{ for } i = 0,1,\ldots,s \text{ and } j = k-n,\ldots,s.$$

Since there are no backorders, $P_{ij}(r) = P_{0j}(r)$ for $i = k-n,\ldots,-1$ and all j.

The problem may now be formulated as a linear programming problem [1] with decision variables y_{ir} where y_{ir} = P{state = i and repair rate = r}. For a given value of s, we formulate the problem of minimizing long run expected average cost of repairs and shortages per unit of time as (P):

$$\text{(P):} \quad \min E[C(s)] = \sum_{i=k-n}^{s} \sum_{r=1}^{R} K_r y_{ir} + \sum_{i=k-n}^{-1} \sum_{r=1}^{R} L_i y_{ir}$$

subject to

$$(1) \quad \sum_{i=k-n}^{s} \sum_{r=1}^{R} y_{ir} = 1,$$

$$(2) \quad \sum_{r=1}^{R} y_{jr} - \sum_{i=k-n}^{s} \sum_{r=1}^{R} y_{ir} P_{ij}(r) = 0 \text{ for } j = k-n,\ldots,s-1,$$

$$(3) \quad y_{ir} \geq 0, \; i=k-n,\ldots,s; \; r=1,2,\ldots R.$$

This problem is of reasonable size, having only R(s+n-k+1) variables and s+n-k+1 constraints, besides the non-negativity constraints. There are s+n-k+1 possible states of the system so there are s+n-k+1 nonzero y_{ir} values in the optimal solution. If $y_{ir} > 0$, r is the repair rate to be used when there are i on hand. There is exactly one positive y_{ir} for each i so the optimal repair policy is deterministic [1].

Problem (P) can be solved by the simplex method. Alternatively, it can be solved by a policy-improvement algorithm, which is a good choice for this problem since many possible policies can be rejected as clearly non-optimal, reducing the potential number of policies to be examined. In particular, no strategy with fast repair for i=q and slow repair for i=q-1 would ever be optimal, since fast repair costs more than slow repair and there is greater chance of stockout at next overhaul in state i-q-1 than for state i=q. Further, if the criterion is to minimize expected total discounted costs, the method of successive approximations [1] may be used. This method has the advantage of never requiring solution of a system of simultaneous equations. On the other hand, the optimal policy will not necessarily be reached in a finite number of iterations.

A separate optimal repair strategy must be determined for each value of s. The total expected repair and stockout cost $E[C(s)]^*$ is added to the holding cost Hs to get total expected costs. The optimal value of s and its associated strategy is the one that minimizes total expected costs.

4. ILLUSTRATIVE EXAMPLE

Consider a machine containing a 4-out-of-6 redundant system. We assume that the machine always arrives for overhaul functioning, so that no more than 2 components have failed.[1] The other parameters are as follows:

$$a = 0.05,\ R = 2,\ b_1 = 0.2,\ b_2 = 0.6,\ L_{-1} = \$500,\ L_{-2} = \$800,\ K_1 = \$50,\ K_2 = \$75.$$

The results are summarized in Table 1, for $s \leq 4$. For each positive value of s, the optimal strategy is to use the fast repair rate only when there are no spares on hand.

As noted earlier, holding costs have not yet been considered. The daily holding costs, Hs, must be added to $E[C(s)]^*$ to get total costs, TC(s). The range of H values for which each s value is optimal may be calculated easily by finding the breakeven points where $E[C(s)]^* + Hs = E[C(s+1)]^* + H(s+1)$. The resulting intervals are shown in Table 2.

[1]*This assumption should be verified by calculating the probability that the number failed exceeds n-k. In our example, this probability is only 0.002.*

TABLE 1

Linear Programming Solutions

s	y_{ik}^*	Optimal Strategy	$E[C(s)]^*$
0	$v_{-2} = .0305$ $v_{-1} = .2321$ $v_0 = .7351$	Emergency repair only, no spares on hand. Probabilities of stockout are merely the v_m measures.	\$140.45
1	$y_{-22} = .0064$ $y_{-12} = .0608$ $y_{02} = .3075$ $y_{11} = .6254$	Fast repair for $i = -2,-1,0$, slow repair for $i \geq 1$.	\$ 94.84
2	$y_{-22} = .0017$ $y_{-12} = .0201$ $y_{02} = .1346$ $y_{11} = .4422$ $y_{21} = .4015$	Fast repair for $i = -2,-1,0$, slow repair for $i \geq 1$.	\$ 65.33
3	$y_{-22} = .0005$ $y_{-12} = .0070$ $y_{02} = .0557$ $y_{11} = .2340$ $y_{21} = .4003$ $y_{31} = .3026$	Fast repair for $i = -2,-1,0$, slow repair for $i \geq 1$.	\$ 55.48
4	$y_{-22} = .0002$ $y_{-12} = .0024$ $y_{02} = .0212$ $y_{11} = .1047$ $y_{21} = .2597$ $y_{31} = .3679$ $y_{41} = .2440$	Fast repair for $i = -2,-1,0$, slow repair for $i \geq 1$.	\$ 51.95

TABLE 2

H and TC(s) Intervals

s	H Values For Which s Is Optimal	Resulting TC(s)
0	$H \geq \$45.61$	$TC(0) = \$140.45$
1	$\$29.51 \leq H < \$45.61$	$124.32 \leq TC(1) < \$140.45$
2	$\$9.85 \leq H < \$29.51$	$75.18 \leq TC(2) < \$124.32$
3	$\$3.53 \leq H < \$9.85$	$59.01 \leq TC(3) < \$\ 75.18$
4	$H \leq \$3.53$	$TC(4) \leq \$59.01$

Thus, by considering holding costs last, a simple sensitivity analysis for H is apparent. If holding costs are difficult to estimate exactly, it may be easier to decide in which interval the H value lies instead. Likewise, if H increases or decreases, Table 2 indicates whether a change in s is required.

The holding cost needn't be linear. A non-linear function H(s) presents no problem since TC(s) is enumerated.

5. EXTENSIONS

In this section we describe the required changes in the model when (a) component failure rates are increasing rather than constant and preventive replacement is necessary, and (b) when there is congestion possible in the repair shop because there is only a single repairman.

(a) When component failure rates increase over time, we assume that the component must be replaced preventively after M cycles. Then the probability of component failure between overhauls will increase with the number of overhaul cycles completed. Letting the number of cycles completed be i, the probability of component failure on cycle i+1 is denoted by a_{i+1}, $i=0,1,\dots,M-1$. Then the steady-state component age distribution may be determined by finding steady-state probabilities for a simple discrete Markov chain with transition probabilities p_{ij} defined as follows:

$p_{i,i+1} = 1-a_{i+1}$ for $i=0,1,\dots,M-1$;

$p_{i0} = a_{i+1}$ for $i=0,1,\dots,M-1$;

$p_{Mj} = p_{0j}$ for $j=0,1$, and $p_{ij} = 0$ elsewhere.

Here state 0 means the part must be replaced because of failure, and state $i=1,\dots,M$ means the part has successfully gone through i cycles. Because the matrix is sparse, closed-form expressions for the steady-state probabilities are available. In particular, given a set of $\{a_i\}$ probabilities, the steady-state probability that, at an overhaul, a component will need to be replaced because of failure is given by $\Pi_o=(1-\bar{a}_1\bar{a}_2\dots\bar{a}_M)/(1+\bar{a}_1+\bar{a}_1\bar{a}_2+\dots+\bar{a}_1\bar{a}_2\dots\bar{a}_{M-1})$ while the probability of preventive replacement is given by $\Pi_M=a_1\dots a_M/(1+\bar{a}_1\bar{a}_2\dots\bar{a}_{M-1})$ where $\bar{a}_i=1-a_i, i=1,2,\dots,M$. Thus the probability that a particular component will need to be replaced at an overhaul is given by $\Pi=\Pi_o+\Pi_M$. For a k-out-of-n system, the probability v_m that exactly m parts are demanded is now $v_m = \binom{n}{m} \Pi^m (1-\Pi)^{n-m}$. The rest of the analysis remains the same.

If M is not specified but a constraint on mission reliability exists, that is, the probability that the system fails between overhauls cannot exceed some small number, α, then M may be determined. Letting $\Pi_o(M)$ be the probability that a part fails between overhauls when replacement occurs after M cycles, the optimal M will be the largest value satisfying

$$\sum_{j=n-k+1}^{n} \binom{n}{j} \Pi_o(M)^j (1-\Pi_o(M))^{n-j} \leqslant \alpha,$$ which may be found by a simple search,

since failure probability increases with M.

(b) The second generalization involves the repair times. The earlier assumption that service times were independent, identically distributed exponential random variables implies that there are an unlimited number of repairmen, that a failed unit will begin undergoing repair immediately. In many contexts, this assumption is not realistic, since congestion will appear when there are few servers and many units requiring repair, and congestion will affect the repair times. As an alternative, consider an M/M/1 queuing system with a single server, exponential repair times, and first come, first served discipline. Then $Z(t,u)$ no longer has a binomial distribution. Instead, repaired parts will leave the system according to a Poisson distribution until no further items remain. We have

$$Z(t,u) = (e^{-\mu}\mu^t)/t! \quad \text{if } 0 \leqslant t < u$$

$$\text{and } Z(u,u) = \sum_{k=u}^{\infty} (e^{-\mu}\mu^k)/k! = 1 - \sum_{k=0}^{u-1} Z(k,u).$$

Since the states of the Markov chain are the number of items on hand, it is clear that the Markov assumption that the next state of the system depends only on the present state remains valid.

The assumption of the model that there is a choice of repair rates may be interpreted to be a choice in the number of repairmen assigned to the repair team or to overtime options for an existing team. The model could be extended to the multi-server case but this would complicate the calculation of Z and the P_{ij}'s and we do not consider that case here.

CONCLUSION

We have applied Markov decision process techniques to a new problem, simultaneously finding the optimal initial number of repairable spares and the optimal repair strategy for maintenance center overhauls of redundant systems. The model requires solution of a reasonable number of linear programming problems by the simplex method or a policy-improvement algorithm. The required assumptions are not very restrictive, extensions are possible, and few parameters must be estimated.

REFERENCES

1. Heyman, D.P. and Sobol, M.J., Stochastic Models in Operations Research, Vol. II, Stochastic Optimization, McGraw-Hill, Inc., New York, 1984.

2. Nahmias, S., "Managing Reparable Inventory Systems: A Review," Chapter 12 in Multi-Level Production/Inventory Control Systems: Theory and Practice, L. B. Schwartz (Ed.), TIMS Studies in the Management Sciences, North-Holland, Amsterdam, 16, 253-278 (1981).

3. Lawrence, S. H. and Schaefer, M. K., "Optimal Maintenance Center Inventories for Fault-Tolerant Repairable Systems", Journal of Operations Management, 4, 175-181 (1984).

4. Schaefer, M. K., "A Multi-Item Maintenance Center Inventory Model for Low-Demand Repairable Items", Management Science, 29, 1062-1068 (1983).

5. Schaefer, M. K., "Optimal Inventories for Overhaul of Repairable Redundant Systems: A Markov Decision Model," Stochastic Failure Models, Replacement and Maintenance Policies, and Accelerated Life Testing, M. Abdel-Hameed, (Ed.) Academic Press, 1984.